Concordia University
Theoretical Computer Science
Solution to Assignment 1
Winter 2015

---

1. The *reversal* of a string w, denoted by w^R, is the string "spelled backwards". For example $(cat)^R = tac$. Reversal is defined inductively as follows:

$$\epsilon^R = \epsilon, (wa)^R = a(w^R).$$

(Here a is a symbol in the alphabet Σ, and w is a string in Σ^*.)

Let $u, v \in \Sigma^*$. Prove that

$$(uv)^R = v^R u^R.$$

Hint: Use induction on $|v|$.

Solution:

By induction on length of v,

Basis: if $|v| = 0$ then $v = \epsilon$, and $(uv)^R = (u\epsilon)^R = (u)^R = u^R = \epsilon u^R = \epsilon^R u^R$.

IH: If $|v| = n$ we have $(uv)^R = v^R u^R$

IS: Now we show that if $|v| = n + 1$ then the claim in the inductive hypothesis is also true.

$|v| = n + 1 \Rightarrow v = xa$, where $|x| = n$ and $a \in \Sigma$.

Then $(uv)^R = (uxa)^R = a(ux)^R = a(x^R u^R) = (ax^R)u^R = (xa)^R u^R = v^R u^R$.

2. Let $\Sigma = \{a, b\}$. For each of the languages below, give an example of a string in the language, and a string not in the language.

   (a) $L_1 = \{w \in \Sigma^* : w = uu^R u, \text{ for some } u \in \Sigma^2\}$

   Solution:

   $w = aaaaaa \in L_1$, since $u = aa \Rightarrow w = uu^R u$

   $w = aa \notin L_1$, since $\nexists u \in \Sigma^2$ such that $w = aa = uu^R u$

   (b) $L_2 = \{w \in \Sigma^* : ww = www\}$

   Solution:

   $w = \epsilon \in L_2$, since $ww = \epsilon = www$

   $w = aa \notin L_2$, since $ww = aaaa \neq aaaaaa = www$

   (c) $L_3 = \{w \in \Sigma^* : uvw = wvu, \text{ for some } u, v \in \Sigma^*\}$.

   Solution:

   $w = ab \in L_3$, since $u = ab, v = aba \Rightarrow uvw = ababaab = wvu$.

   $L_3 = \Sigma^*$, since for any w, we can choose $u = w$ and $v = \epsilon$.

   (d) $L_4 = \{w \in \Sigma^* : www = uu, \text{ for some } u \in \Sigma^*\}$.

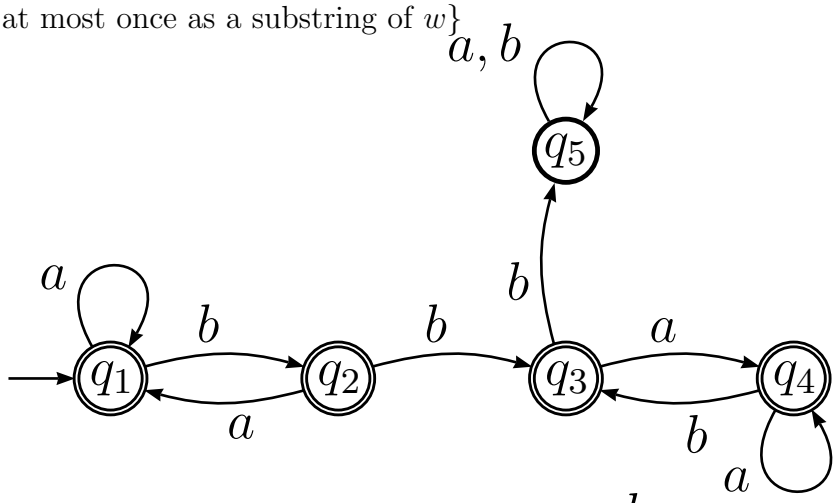
   Solution:

   $w = aa \in L_4$, since $u = aaa \Rightarrow www = aaaaaa = uu$.

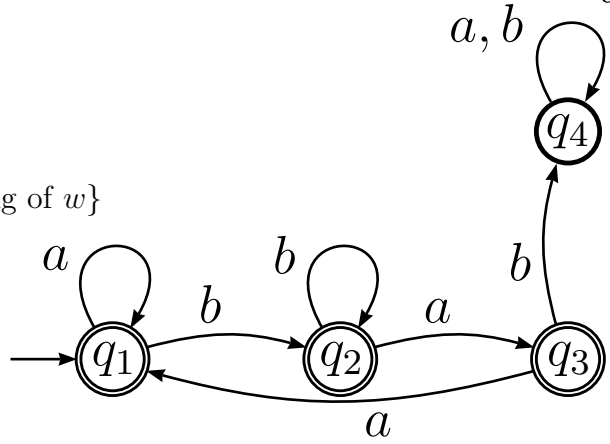
   $w = a \notin L_4$, since $www = aaa$ and $\nexists u$ such that $uu = aaa$.

3. Construct a DFA for each of the following languages.

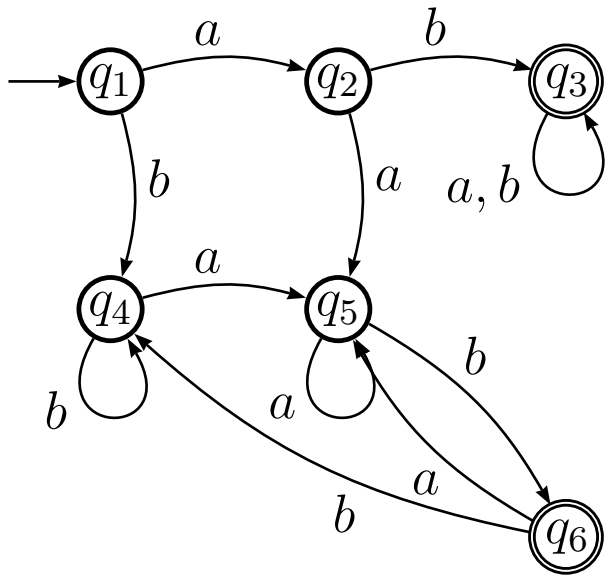
(a) $\{w \in \{a,b\}^* : bb$ appears at most once as a substring of $w\}$



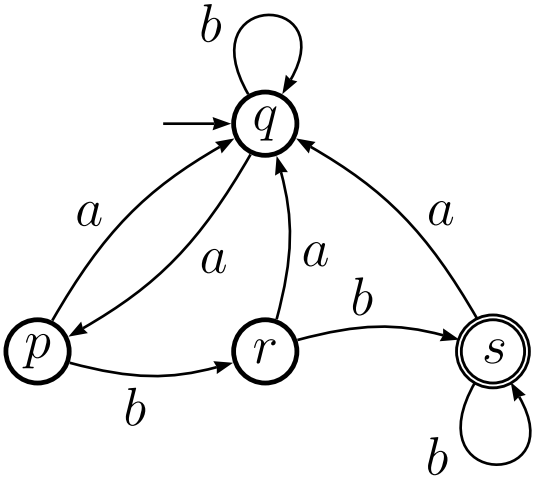
(b) $\{w \in \{a,b\}^* : bab$ is not a substring of $w\}$



(c) The set of strings that either begin or end (or both) with ab.



(d) $\{w \in \{a,b\}^* : w$ contains an odd number of $a's$ and ends in at least two $b's\}$



Give your DFA's as transition diagrams.

	a	b
$\rightarrow q$	p	q
p	q	r
r	q	s
$*s$	q	s

4. Let $L = \{w \in \{0,1\}^* \ : \ w$ has an odd no. of 1's $\}$, and let A be the DFA with tabular representation:

A	0	1
$\rightarrow p$	p	q
$\star\ q$	q	p

Prove that $L = L(A)$. *Hint:* Do the $L(A) \subseteq L$ part of the proof by induction on the the length of the string processed by A. You need a mutual induction with a claim for state p and a claim for state q.

Solution:

To prove $w \in L(A) \Rightarrow w \in L$ we use an induction on length of w. We claim that

(1) If $\hat{\delta}(p, w) = p$, then w has even number of 1's.

(2) If $\hat{\delta}(p, w) = q$, then w has odd number of 1's.

Basis: $|w| = 0$ then $w = \epsilon$, so obviously (1) is true. Also, it is clear that $\hat{\delta}(p, \epsilon) = p \neq q$, so (2) is vacuously true.

IH: (1) and (2) are true for any string w of length n.

IS: Now prove (1) and (2) for $w = xa$, where $|x| = n$ and $a \in \Sigma$.

(1): If $\hat{\delta}(p, xa) = p$ then $\hat{\delta}(p, x)$ is p and $a = 0$ or q and $a = 1$, as can be seen from the transition diagram of A.

If $\hat{\delta}(p, x) = p$ then by the IH (1), the string x has even number of 1's. Therefore $w = x0$ which has even number of 1's.

If $\hat{\delta}(p, x) = q$ then by the IH (2), the string x has odd number of 1's, so $w = x1$ has even number of 1's.

(2): If $\hat{\delta}(p, xa) = q$ then $\hat{\delta}(p, x)$ is p and $a = 1$ or q and $a = 0$, as can be seen from the transition diagram of A.

If $\hat{\delta}(p, x) = p$ then by the IH (1), the string x has even number of 1's. Therefore $w = x1$ which has odd number of 1's.

If $\hat{\delta}(p, x) = q$ then by the IH (2), the string x has odd number of 1's, so $w = x0$ has odd number of 1's. This completes induction.

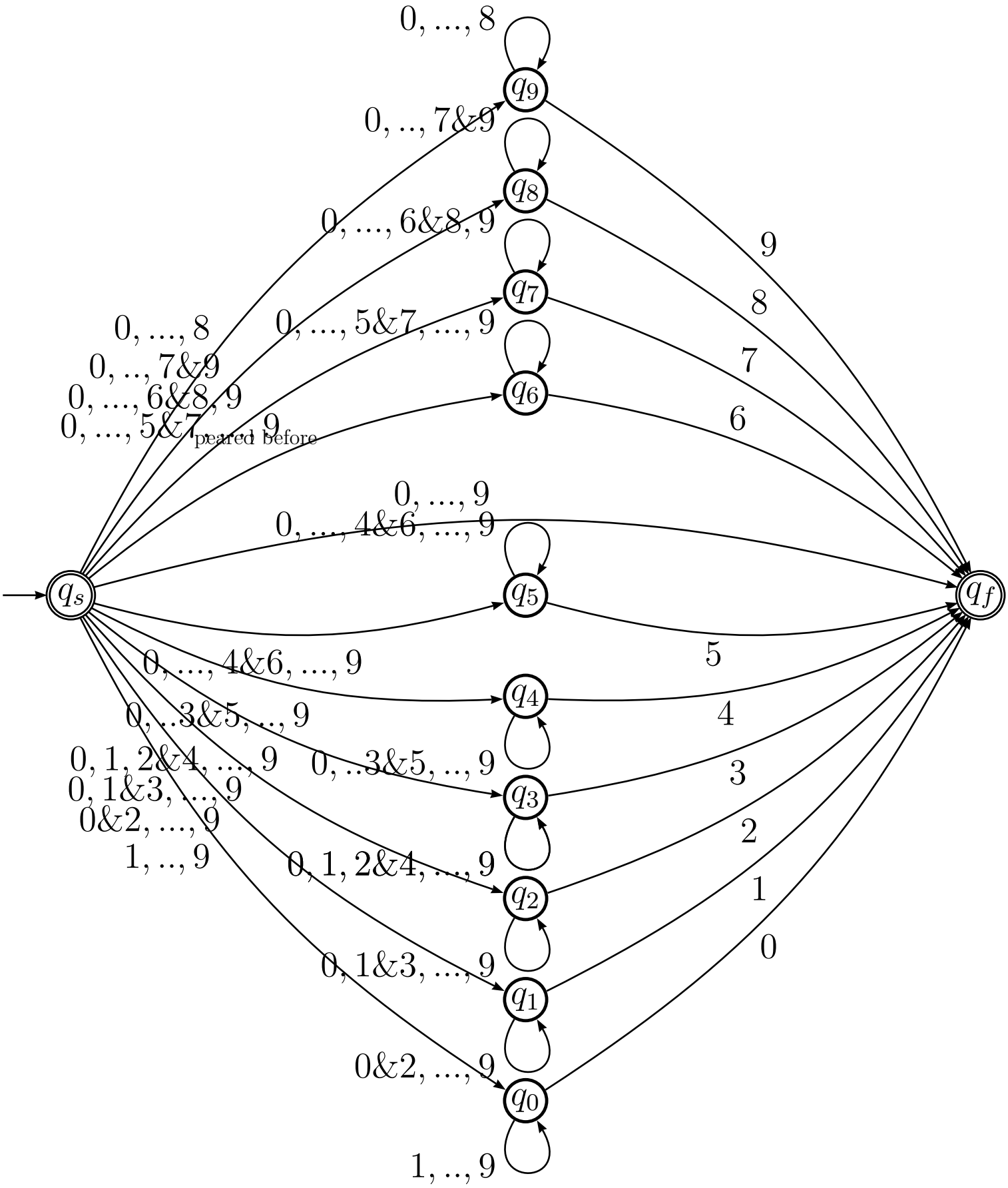
For the other direction, we have:

$$\begin{aligned} w \notin L(A) &\Rightarrow \hat{\delta}(p, w) = p \\ &\Rightarrow w \text{ has even number of 1's} \\ &\Rightarrow w \notin L \end{aligned}$$

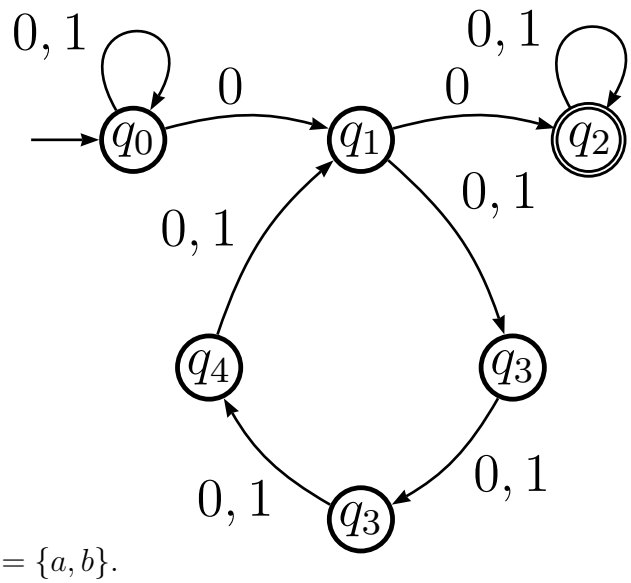
So it never ends in a final state with an even number of 1's. Therefore, this direction is also proved, that is $L \subseteq L(A)$.

5. Construct an NFA for each of the following languages.

(a) The set of strings over $\{0, 1, \ldots, 9\}$, such that the final digit has not appeared before

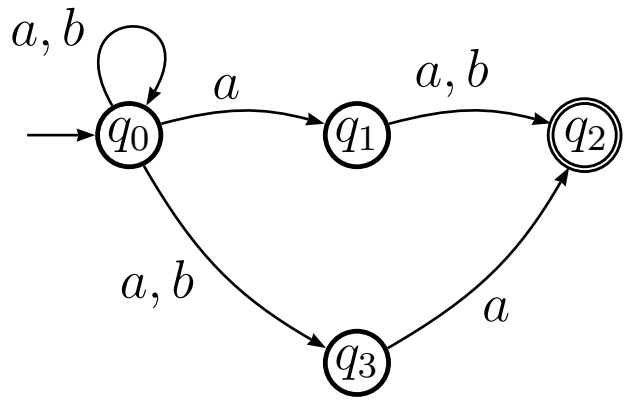


(b) The set of strings over $\{0, 1\}$, such that there are two 0's separated by a number of positions that is a multiple of 4. Note that 0 is an allowable multiple of 4.

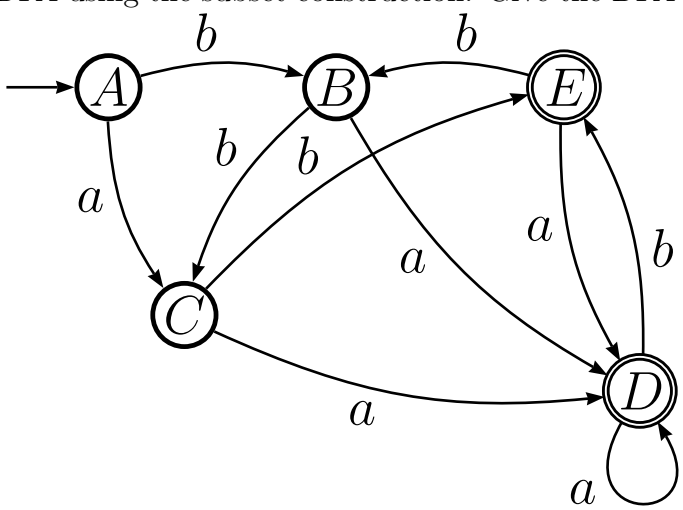


6. Let $\Sigma = \{a, b\}$.

(a) Construct an NFA that accepts the strings in Σ^* where at least one of the last two symbols is an a.



(b) Convert your NFA to a DFA using the subset construction. Give the DFA



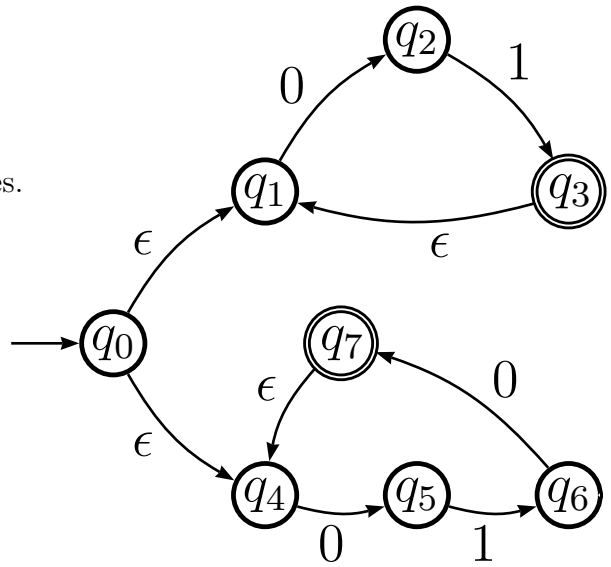
both in tabular form and as a transition diagram.

	a	b
$\rightarrow A$	C	B
B	D	C
C	D	E
$*D$	D	E
$*E$	D	B

$$A = \{q_0\}$$
$$B = \{q_0, q_2\}$$
$$C = \{q_0, q_1, q_2\}$$
$$D = \{q_0, q_1, q_2, q_3\}$$
$$E = \{q_0, q_2, q_3\}$$

7. Let $\Sigma = \{0, 1\}$. Design ϵ-NFA's for the following languages.

(a) The set of string that consists of either 01 repeated one or more times or 010 repeated one or more times.



(b) The set of strings such that at least one of the last ten positions is a 1.