Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Mathematics	203	All
Examination	**Date**	**Pages**
Final	April/May 2006	3
Instructors		**Course Examiner**
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Special Instructions

▷ Calculators are **not** allowed.

MARKS

[10] **1.** (a) Suppose $f(x) = \sqrt{x-1}$ and $g(x) = 1 + \left(\dfrac{x}{1+x^2}\right)^2$. Find $f \circ g$, $g \circ f$ and $f \circ f$.

(b) Find the inverse of the function $f(x) = \ln(1 + x^3)$. Determine the domain and range of f and f^{-1}.

[10] **2.** Evaluate the limits:

(a) $\displaystyle\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{2x^2-8}$ (b) $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{2x^2+1}}{x+1}$

Do not use l'Hopital's rule.

[12] **3.** (a) Consider the function $f(x) = \dfrac{|x+1|}{x^2+x}$.

Calculate both one-sided limits at the point(s) where the function is undefined.

(b) Find parameters a and b such that the function

$$f(x) = \begin{cases} -x^2 - 1, & \text{if} \quad x \le 0 \\ ax + b, & \text{if} \quad 0 < x \le 2 \\ \frac{2}{x}, & \text{if} \quad x > 2 \end{cases}$$

will be continuous at every point. Sketch the graph of this function.

[12] **4.** Find derivatives of the functions (do not simplify the answer):

(a) $f(x) = (x^3 + 2x + 5)\sin 2x\,;$

(b) $f(x) = \ln^2\left(1 + \cos^2 5x\right);$

(c) $f(x) = \dfrac{\arccos^3 x}{\sqrt{1 - x^2}}\,;$

(d) $f(x) = (1 + x^2)^{\arctan x}$ (use logarithmic differentiation).

[12] **5.** Given the function $f(x) = \sqrt{x^2 + 24}\,,$

(a) Use appropriate differentiation rules to find the derivative of the function.

(b) Use the definition of derivative to verify (a).

(c) Find the linear approximation of the function at $x_0 = 1$.

(d) Use the linear approximation above to approximate $\sqrt{28}\,$.

[18] **6.** (a) The equation of a curve defined implicitly is $y^2 \cos x = xy^5 + y + 2$. Verify that the point $(0, -1)$ belongs to the curve. Find an equation of the tangent line to the curve at this point.

(b) Let $f(x) = \dfrac{12 + x^3}{2x^3}$. Find $f^{(n)}(x)$.

(c) Use l'Hopital's rule to evaluate $\displaystyle\lim_{x \to 0} \frac{\sin^2 x}{x \ln(1 + 3x)}\,.$

[10] **7.** (a) A particle is moving along the plane curve $y^2 - 6x^4 = y$. At the moment when $x = -1$ the x-coordinate is increasing at a rate of 5 cm/sec. If the y-coordinate is negative at this moment, is y increasing or decreasing? How fast?

(b) A rectangle $ABCD$ has sides parallel to the coordinate axes and point A is located at the origin. A point C belongs to the graph of the exponential function $y = e^{10x}$ and has a negative x coordinate. Find the coordinates of the point C that maximize the area of the rectangle.

[16] **8.** Given the function $f(x) = \dfrac{x^2}{x^2 - 4}$,

(a) Find the domain and check for symmetry. Find asymptotes (if any).

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function.

[5] **Bonus Question**

Given the equation $10x^3 + x = 10$,

(a) Show that there is a root between $\frac{1}{2}$ and 1.

(b) Show that the equation has exactly one root.