Midterm MATH 251, October 19, 2012

Justify all answers

Problem 1 [5 pt] Are the following subsets of $\mathbb{R}^5$ subspaces? **Justify your answer**.

$$(\mathbf{a}) \quad W_1 := \left\{ \langle a_1, a_2, a_3, a_4, a_5 \rangle : \quad a_1^2 + a_2 = 0, \;\; a_4 + a_5 = 0 \right\} \tag{1}$$

$$(\mathbf{b}) \quad W_2 := \{ \langle a_1, a_2, a_3, a_4, a_5 \rangle : \quad 2a_2 + a_3 = 0, \quad a_1 + 3a_5 = 0 \} \tag{2}$$

Problem 2 [5 pt] Find a basis for the following subspace of $\mathbb{R}^5$

$$W := \{ \langle a_1, a_2, a_3, a_4, a_5 \rangle : \quad a_1 - 2a_2 = 0 \;, \quad a_3 + a_4 + a_5 = 0 \} \tag{3}$$

Problem 3 [5 pt] Let $V = Mat_{2\times 3}(\mathbb{R})$ and consider the following subspaces:

$$W_1 := \left\{ \begin{bmatrix} a & 2a & a+b \\ c & d & 0 \end{bmatrix} \quad a, b, c, d \in \mathbb{R} \right\} , \qquad W_2 := \left\{ \begin{bmatrix} f & -f & g \\ e & e & \ell \end{bmatrix} \quad e, f, g, \ell \in \mathbb{R} \right\} \tag{4}$$

Find the dimensions of $W_1, \; W_2, \; W_1 + W_2, \; W_1 \cap W_2$. Verify that

$$\dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) = \dim(W_1 + W_2) \tag{5}$$

Problem 4 [5 pt] Let $T : V \to W$ be a linear transformation. Let $\{\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_k\}$ be a linearly independent subset of $R(T)$, (the range of T). Prove that if $\underline{v}_1, \ldots, \underline{v}_k \in V$ are pre-images of the $\underline{w}_j$'s, that is, $T\underline{v}_j = \underline{w}_j$ for $j = 1, \ldots, k$, then the set $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ is linearly independent.

Problem 5 [5 pt] Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(\langle a_1, a_2 \rangle) = \langle 2a_1, a_1 + a_2, 5a_1 - a_2 \rangle$. Let β be the standard basis of $\mathbb{R}^2$ and $\alpha = (\langle 1, 1 \rangle, \langle 1, -1 \rangle)$ another basis of $\mathbb{R}^2$. Let $\gamma = (\langle 1, 1, 1 \rangle, \langle 0, 1, 1 \rangle, \langle 0, 0, 1 \rangle)$ be a basis of $\mathbb{R}^3$. Compute

$$[T]_\beta^\gamma, \qquad [T]_\alpha^\gamma \;. \tag{6}$$

Problem 6 [5 pt] Consider the transformation $T : \mathcal{P}_3 \to \mathcal{P}_3$ where $\mathcal{P}_3$ the finite dimensional vector space consisting of polynomials of degree up to 3.

$$T(p(x)) = x^2 p''(x) + p(x-1) \tag{7}$$

Note: here $p(x-1)$ means the shift of variable, for example if $p(x) = x^2 + 2$ then $p(x-1) = (x-1)^2 + 2 = x^2 - 2x + 3$.

1. Show that T is linear;
2. Find $[T]_\beta$ where $\beta = (1, x, x^2, x^3)$ is the standard ordered basis of $\mathcal{P}_3$.

Problem 7 [Bonus 3 pt] Let $T : V \to W$ and $U : W \to Z$ be two linear transformations between the indicated vector spaces V, W, Z. Prove that $\mathbf{N}(T) \subseteq \mathbf{N}(UT)$, where $\mathbf{N}(T)$, $\mathbf{N}(UT)$ denote the kernels (null-spaces) of the indicated transformations. Give an example where the inclusion is strict.

Solution to Problem 1 The set W_1 is not a subspace because the sum of two vectors in it is not necessarily still in the same space For example $\underline{v}_1 =< 1, -1, 0, 0, 0 >\in W_1$ and also $\underline{v}_2 =< 2, -4, 0, 0, 0 >\in W_1$, but $\underline{v}_1 + \underline{v}_2 =< 3, -5, 0, 0, 0 >$ is not because $3^2 - 5 = 4 \neq 0$ and thus it is not in W_1.
The set W_2 is a subspace:

- $\underline{0} \in W_2$ because $2(0) + 0 = 0$ and $0 + 3(0) = 0$;
- if $\langle a_1, a_2, a_3, a_4, a_5\rangle, \langle b_1, b_2, b_3, b_4, b_5\rangle \in W_2$ then their sum satisfies the conditions since

$$2(a_2+b_2)+(a_3+b_3) = \overbrace{2a_2 + a_3}^{=0} + \overbrace{2b_2 + b_3}^{=0} = 0 \qquad (a_3+b_3)+(a_4+b_4)+(a_5+b_5) = \overbrace{a_3 + a_4 + a_5}^{=0} + \overbrace{b_3 + b_4 + b_5}^{=0} = 0 \tag{8}$$

- if $c \in \mathbb{R}$ and $\underline{v} = \langle a_1, a_2, a_3, a_4, a_5\rangle \in W_2$ then $c\underline{v} = \langle ca_1, ca_2, ca_3, ca_4, ca_5\rangle \in W_2$ because

$$2(ca_2) + (ca_3) = c(\overbrace{2a_2 + a_3}^{=0}) \ , \quad ca_3 + ca_4 + ca_5 = c(\overbrace{a_3 + a_4 + a_5}^{=0}) = 0 \tag{9}$$

□

Solution to Problem 2 Since $a_1 = 2a_2$ and $a_5 = -a_3 - a_4$ then any vector in W has the form

$$< 2a_2, a_2, a_3, a_4, -a_3 - a_4 >= a_2 < 2, 1, 0, 0, 0 > +a_3 < 0, 0, 1, 0, -1 > +a_4 < 0, 0, 0, 1, -1 >; \tag{10}$$

so the three indicated vectors span W. They are linearly independent because setting the lhs to zero implies $a_2 = 0$, $a_3 = 0$, $a_4 = 0$ by looking at the entries 1,2,3. The dimension of the space is 3 and the basis is for example the collection of the three vectors above. □

Solution to Problem 3 A matrix in the sum $W_1 + W_2$ has the form

$$\begin{bmatrix} a+f & 2a-f & a+b+g \\ c+e & d+e & \ell \end{bmatrix} \tag{11}$$

We claim that any matrix in $V = Mat_{2\times 3}$ can be expressed in the above form. To see it let $M = \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix}$. Equating the entries we have

$$\begin{bmatrix} a+f & 2a-f & a+b+g \\ c+e & d+e & \ell \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \Rightarrow \begin{cases} a+f = A \\ 2a-f = B \\ a+b+g = C \\ c+e = D \\ d+e = E \\ \ell = F \end{cases} \tag{12}$$

$$\begin{cases} a = \frac{A+B}{3} \\ f = \frac{2A-B}{3} \\ g = C - \frac{A+B}{3} - b \\ c = D - e \\ d = E - e \\ \ell = F \end{cases} \tag{13}$$

where b, e can be arbitrary. Thus the $\dim(W_1 + W_2) = \dim V = 6$. On the other hand $M \in W_1$

$$M = a(E^{11} + 2E^{12} + E^{13}) + bE^{13} + cE^{21} + dE^{22} \tag{14}$$

and we can see that the four matrices multiplying a, b, c, d are independent (setting $M = 0$ gives $a = 0$ by looking at the 11 entry, hence $b = 0$, $c = 0, d = 0$ looking at the other entries. Thus $\dim W_1 = 4$. Similarly $M \in W_2$

$$M = f(E^{11} - E^{12}) + gE^{13} + e(E^{21} + E^{22}) + \ell E^{32} \tag{15}$$

and the same argument shows that these matrices are independent. Hence $\dim W_2 = 4$.
The intersection. We have to equate

$$\begin{bmatrix} a & 2a & a+b \\ c & d & 0 \end{bmatrix} = \begin{bmatrix} f & -f & g \\ e & e & \ell \end{bmatrix} \tag{16}$$

from which we have $a = 0, f = 0$, $c = d = e$, $\ell = 0$, $b = g$. So the matrices in the intersection are of the form

$$\begin{bmatrix} 0 & 0 & b \\ c & c & 0 \end{bmatrix} \tag{17}$$

and the dimension is 2.

Thus

$$4 + 4 - 2 = 6 \tag{18}$$

as expected. □

Solution to Problem 4 Since $\underline{w}_k$ are independent then the only solution to

$$\underline{0}_W = \sum_{j=1}^{k} c_j \underline{w}_j \tag{19}$$

is the trivial solution. Now, consider the similar equation

$$\underline{0}_V = \sum_{j=1}^{k} c_j \underline{v}_j \tag{20}$$

Applying T to both sides we have

$$\underline{0}_W = T\underline{0}_V = T\left(\sum_{j=1}^{k} c_j \underline{v}_j\right) \overset{\text{by linearity}}{=} \sum_{j=1}^{k} c_j T\underline{v}_j = \sum_{j=1}^{k} c_j \underline{w}_j \tag{21}$$

Since the only solution of eq. (19) is the trivial one, it implies that all c_j's are zero. Thus the equation (20) implies $c_1 = 0 = \ldots = c_k$ and hence $\underline{v}_j$'s are also independent. □

Solution to Problem 5 We have

$$T<1,0>=<2,1,5>=2<1,1,1>-<0,1,1>+4<0,0,1>; \tag{22}$$
$$T<0,1>=<0,1,-1>=0<1,1,1>+<0,1,1>-2<0,0,1>; \tag{23}$$
$$[T]_\beta^\gamma=\begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 4 & -2 \end{bmatrix} \tag{24}$$

$$T<1,1>=<2,2,4>=2<1,1,1>+0<0,1,1>+2<0,0,1>; \tag{25}$$
$$T<1,-1>=<2,0,6>=2<1,1,1>-2<0,1,1>+6<0,0,1>; \tag{26}$$
$$[T]_\beta^\gamma=\begin{bmatrix} 2 & 2 \\ 0 & -2 \\ 2 & 6 \end{bmatrix} \tag{27}$$

□

Solution to Problem 6 The map is linear; $T0=0$ (the shift of the polynomial $p(x)=0$ is $p(x-1)=0$ as well)

$$T((p+q)(x))=x^2(p''(x)+q''(x))+p(x-1)+q(x-1)=x^2p''(x)+p(x-1)+q(x-1)=T(p(x))+T(q(x)) \tag{28}$$

$$T(\lambda p(x))=x^2\lambda p''(x)+\lambda p(x-1)=\lambda(x^2p''(x)+p(x-1))=\lambda T(p(x)) \tag{29}$$

Then:

$$T(1) = x^2(1)''+1 = 1+0x+0x^2+0x^3; \tag{30}$$
$$T(x) = x^2(x)''+(x-1) = -1+x+0x^2+0x^3; \tag{31}$$
$$T(x^2) = x^2(x^2)''+(x-1)^2=2x^2+x^2-2x+1 = 1-2x+3x^2+0x^3 \tag{32}$$
$$T(x^3) = x^2(x^3)''+(x-1)^3=6x^3+x^3-3x^2+3x-1 = -1+3x-3x^2+7x^3 \tag{33}$$
$$\tag{34}$$

Thus

$$[T]_\beta=\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 7 \end{bmatrix} \tag{35}$$

□

Solution to Problem 7 If $\underline{v}\in \mathbf{N}(T)$ then

$$UT(\underline{v}) \overset{\text{by def.}}{=} U(T(\underline{v})) \overset{\underline{v}\in\mathbf{N}(T)}{=} U(\underline{0}_W) \overset{\text{by linearity of } U}{=} \underline{0}_Z \tag{36}$$

Thus $\underline{v} \in \mathbf{N}(UT)$ and hence any vector in the kernel of T is in the kernel of UT and the inclusion is proved. To show that the inclusion can be strict, consider the example where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity map (with trivial kernel)

$$T\underline{v} = \underline{v} \ , \quad \forall \underline{v} \in \mathbb{R}^2 \ , \tag{37}$$

and $U : \mathbb{R}^2 \to \mathbb{R}^3$ to be the zero transformation (i.e. $U(\underline{w}) = \underline{0}_Z$) Then $\mathbf{N}(T) = \{\underline{0}_{\mathbb{R}^2}\} \subset \mathbf{N}(UT) = \mathbb{R}^2$. $\square$